ARTICLE NO. 80-4057

J80-126

Integral Analysis for the Interaction of Radiation with Conduction in a Half Space

90003
90005
90008

L. T. Yeh*
Babcock & Wilcox Company, Barberton, Ohio
and
B.T.F. Chung†
The University of Akron, Akron, Ohio

A theoretical analysis of combined transient conduction and radiation is developed for a gray, optically thick semi-infinite medium with temperature dependent properties and with an arbitrary time dependent surface temperature. The heat flux due to radiation in the solid is considered to be a diffusion process and the Rosseland's equation for local radiative heat flux is employed. The heat balance integral method is adapted. With the aid of this technique, the complex governing equation represented by a nonlinear partial differential equation is reduced to a first order initial value problem. The numerical results for the temperature surface conditions and various values of parameter N which represents the ratio of energy transfer due to conduction to that due to radiation. For illustrative purposes, the surface temperature is assumed to vary either linearly or exponentially with time. The present results under limiting conditions are compared with solutions available in the literature.

Nomenclature

c = heat capacity, ρc_p; J/m^3-K
f_k = function of thermal conductivity
K = thermal conductivity, $W/m-K$
K_R = Rosseland mean absorption coefficient, m^{-1}
ℓ = characteristics length, unit length for the present case, m
m = coefficient of heat capacity
n = index of refraction
n_k = coefficient of variable conductivity
N = parameter representing the relative measure of heat conducted to heat radiated, $K_0 K_R / 4n^2 \sigma T_0^3$
p = profile parameter
q_r = radiative heat flux, W/m^2
t = dimensional time, h
T = dimensional temperature, K
U = dimensional temperature, T/T_0
x = dimensional space coordinate, m
x_1 = dimensionless space coordinate, x/ℓ
α = thermal diffusivity, K/c; m^2/h
δ = penetration depth, m
δ_1 = dimensionless penetration depth, δ/ℓ
σ = Stefan-Boltzmann constant, W/m^2-K^4
τ = dimensionless time, $\alpha_0 t/\ell^2$

Subscripts

0 = at reference condition
s = surface condition

Introduction

TRANSIENT heat transfer under simultaneous conduction and radiation is a complex phenomenon. However, problems of this type arise frequently in engineering applications such as heat transfer at high temperatures through either porous insulating material or the solids that are transparent to infrared radiation. The governing mathematical expression for this type of problem generally involves an integro-differential equation, the solution of which is usually obtained by numerical methods. Heinisch and Viskanta[1] have solved numerically the transient combined conduction and radiation heat transfer in a semi-infinite, gray, optically thick medium with variable properties. The Rosseland approximation for local radiative heat flux was adapted to simplify the problem and the Boltzmann transformation was used to reduce the governing mathematical expression from a nonlinear partial differential equation to a nonlinear ordinary differential equation. The results were in terms of similarity transform parameters. The same problem was later solved by Lardner,[2] who applied Biot's variational principle to obtain an approximate solution. However, no attempt was made by the author to compare the approximate solution to the previous numerical solution of Ref. 1. Reference 2 is of interest due to its simplicity. Unfortunately it contains a number of errors.‡

Both papers mentioned above are limited to the cases with a constant surface temperature boundary condition. A more

Received April 19, 1979; revision received Oct. 18, 1979.

Index categories: Heat Conduction; Radiative Heat Transfer; Thermal Modeling and Analysis.

*Senior Engineer, Technology, FPGD.

†Associate Professor, Dept. of Mechanical Engineering.

‡Using his notations, Eqs. (14) and (15) of Ref. 2 should read

$$D = \tfrac{1}{2}\int (1/K)\dot{H}\cdot\dot{H}\,\mathrm{d}v, \quad Q^* = \tfrac{1}{3}CT_0(1+\gamma/N)$$

The expression for γ should be

$$\frac{8}{3}\,\frac{13-2b}{(11-2b)(10-2b)}$$

instead of

$$\frac{8}{3}\left[\frac{1}{9-2b}+\frac{3}{(10-2b)(11-2b)}\right]$$

and the corrected expressions for the penetration depth are

$$q/\sqrt{\alpha t} = 3.36[1+(1.333/N)]^{1/2} \quad \text{for} \quad b=3$$

$$q/\sqrt{\alpha t} = 3.36[1+(0.45/N)]^{1/2} \quad \text{for} \quad b=0$$

general case involving an arbitrary time dependent temperature of the surface has not yet been investigated. In this work, an analysis of combined transient conduction and radiation is developed for a gray, optically thick semi-infinite medium with temperature dependent properties and with an arbitrary time dependent surface temperature. Due to the inherent nonlinearity of the energy equation and the complicated boundary conditions, the exact analytical solution is not feasible. In the interest of simplicity of the analysis, an approximate analytical technique known as the heat balance integral method[3,4] is adapted in the present work. Although the heat balance integral method has been successfully applied to heat conduction in an opaque medium with a radiative boundary condition,[5,6] the method has not been extended to the solution of combined conduction-radiation problem.

Mathematical Analysis

Consideration is given to an optically thick semi-infinite solid with temperature dependent thermal properties. It is assumed that medium is initially at a uniform temperature, T_i. A step change in a surface temperature is imposed that can be an arbitrary function of time. The heat flux due to radiation in the solid is considered to be diffusion process and the radiative heat flux is approximated accordingly.

The relevant energy equation for the solid together with its initial and boundary conditions is

$$c(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[K(T)\frac{\partial T}{\partial x}-q_r\right] \quad 0\leq x<\infty \tag{1}$$

$$T=T_i \quad \text{at} \quad t=0 \tag{2a}$$

$$T=T_s(t) \quad \text{at} \quad x=0 \tag{2b}$$

$$T=T_i \quad \text{as} \quad x\to\infty \tag{2c}$$

Using Rosseland's approximation[7] for optically thick material, the radiative flux is reduced to:

$$q_r=-\frac{16n^2\sigma T^3}{3K_R}\frac{\partial T}{\partial x} \tag{3}$$

It should be pointed out that this approximation for the radiative heat flux is strictly valid in the interior of the medium, and it does not properly account for radiation transfer near the boundary. Substituting Eq. (3) into Eq. (1) and carrying out the differentiation gives

$$c\frac{\partial T}{\partial t}=\frac{\partial K}{\partial T}\left(\frac{\partial T}{\partial X}\right)^2+\frac{16n^2\sigma T^2}{K_R}\frac{\partial T}{\partial x}+\left(K+\frac{16n^2\sigma T^3}{3K_R}\right)\frac{\partial^2 T}{\partial x^2} \tag{4}$$

The above equation is strongly nonlinear due to the temperature dependence of thermal properties and to the nonlinear nature of radiative heat flux.

To attack this problem, we employ the heat balance method which is similar to the integral approach of von Kármán in boundary layer flow problems. We assume a second degree polynomial temperature profile within the thermal layer δ of the form

$$T=T_i+(T_s-T_i)\left[1-(x/\delta)\right]^2 \tag{5a}$$

where δ is the penetration depth or thermal layer thickness so that

$$\partial T/\partial X=0 \quad \text{at } x=\delta \tag{2d}$$

Since the heat wave does not penetrate beyond the thickness δ, the condition Eq. (2c) can now be replaced by

$$T=T_i \quad \text{at } x=\delta \tag{2e}$$

Equation (5a) satisfies the boundary conditions given by Eqs. (2b), (2d), and (2e).

Integrating Eq. (1) from $x=0$ to $x=\delta$, and making use of Eqs. (2d), (3), and (5a) yields

$$\left[K_s+\frac{16n^2\sigma}{3K_R}T_s^3\right]2(T_s-T_i)\frac{1}{\delta}=\int_0^\delta c\left[\left(1-\frac{x}{\delta}\right)^2\frac{\mathrm{d}T_s}{\mathrm{d}t}+2(T_s-T_i)\left(1-\frac{x}{\delta}\right)\left(\frac{x}{\delta^2}\right)\frac{\mathrm{d}\delta}{\mathrm{d}t}\right]\cdot\mathrm{d}x \tag{6}$$

To carry out the integration of the above expression, the functional relationship for heat capacity must be specified, but there is no restriction on the thermal conductivity function. Let the thermal conductivity and the heat capacity of the medium be the forms of:

$$K=K_0 f_k(T/T_0) \tag{7}$$

$$c=C_0(1+mT/T_0) \tag{8}$$

where K_0 and C_0 are the thermal conductivity and the heat capacity measured at the reference temperature T_0, respectively, $f_k(T/T_0)$ is an arbitrary function of dimensionless temperature.

Substituting Eqs. (7) and (8) into Eq. (6) and making use of Eq. (5b) gives

$$2K_0\left[f_k\left(\frac{T_s}{T_0}\right)+\frac{16n^2\sigma}{3K_RK_0}T_s^3\right]\frac{(T_s-T_i)}{\delta}=C_0\int_0^\delta\left\{\left(1+m\frac{T_i}{T_0}\right)+m\frac{(T_s-T_i)}{T_0}\left(1-\frac{x}{\delta}\right)^2\right]\left(1-\frac{x}{\delta}\right)^2\frac{\mathrm{d}T_s}{\mathrm{d}t}+2\left[\left(1+m\frac{T_i}{T_0}\right)+\frac{m(T_s-T_i)}{T_0}\left(1-\frac{x}{\delta}\right)^2\right](T_s-T_i)\left(1-\frac{x}{\delta}\right)\frac{x}{\delta^2}\frac{\mathrm{d}\delta}{\mathrm{d}t}\right\}\mathrm{d}x$$

or

$$2K_0\left[f_k\left(\frac{T_s}{T_0}\right)+\frac{16n^2\sigma T_s^3}{3K_RK_0}\right]\frac{(T_s-T_i)}{\delta}=c_0\left\{\left[\frac{1}{3}\left(1+m\frac{T_i}{T_0}\right)+\frac{m}{5}\frac{(T_s-T_i)}{T_0}\right]\delta\frac{\mathrm{d}T_s}{\mathrm{d}t}+\left[\frac{1}{3}\left(1+\frac{mT_i}{T_0}\right)(T_s-T_i)+\frac{m}{10T_0}(T_s-T_i)(T_s-T_i)\right]\frac{\mathrm{d}\delta}{\mathrm{d}t}\right\} \tag{9}$$

Nondimensionalize the above equation by using the following new variables

$$\tau=\alpha_0 t/\ell^2 \quad u=T/T_0 \quad \delta_1=\delta/\ell \quad \text{and} \quad N=K_0K_R/4n^2\sigma T_0^3 \tag{10}$$

where ℓ is the characteristic length. For convenience, let ℓ be unit length for the present problem. Using Eqs. (9) and (10) we arrive at the following dimensionless ordinary differential equation for the penetration depth or thermal thickness.

$$2\delta_1\frac{\mathrm{d}\delta_1}{\mathrm{d}\tau}+P(\tau)\delta_1^2=Q(\tau) \tag{11}$$

where

$$P(\tau)=\frac{⅓(1+mU_i)+1/5m(U_s-U_i)}{½[⅓(1+mU_i)+1/10m(U_s-U_i)](U_s-U_i)}\frac{\mathrm{d}U_s}{\mathrm{d}\tau} \tag{12a}$$

and

$$Q(\tau) = \frac{4}{\frac{1}{3}(1+mU_i) + 1/10m(U_s - U_i)} \left[f_k(U_s) + \frac{4}{3N} U_s^3 \right] \tag{13a}$$

Note that Eq. (11) is linear in δ_l^2 and can be integrated exactly with the aid of an integration factor. Then

$$\delta_l^2 \cdot \exp(\int P(\tau)\,d\tau) = \int Q(\tau) \cdot \exp(\int P(\tau)\,d\tau)\,d\tau + A \tag{14}$$

is obtained.

The integration constant A can be determined by the following initial condition

$$\delta_l = 0 \quad \text{at} \quad \tau = 0 \tag{15}$$

This equation is consistent with the initial condition given by Eq. (2a). The temperature distribution in the solid can be readily determined from Eq. (5), once the depth of penetration δ_l is known. Consequently, the heat flux at wall is readily obtained from the differentiation of Eq. (5).

In Eqs. (12-14) the dimensionless surface temperature U_s remains to be specified. Once the functional relationship is given, P and Q can be determined easily.

For the limiting case of constant surface temperature, the solution of Eq. (14) is reduced to

$$\delta_l = \sqrt{\frac{4[f_k(U_s) + 4U_s^3/3N]\tau}{(1+mU_i)/3 + m(U_s - U_i)/10}} \tag{16}$$

The corresponding temperature distribution given by Eq. (5) becomes

$$U = U_i + (U_s - U_i) \left[1 - \frac{\eta}{\sqrt{\dfrac{f_k(U_s) + 4U_s^3/3N}{(1+mU_i)/3 + m(U_s - U_i)/10}}} \right]^2 \tag{17}$$

where

$$\eta = x_l / 2\sqrt{\tau} \tag{18}$$

Results and Discussion

As can be seen in the analysis, the complicated nonlinear boundary value problem represented by Eqs. (1) and (2) is reduced to a simple initial value problem given by Eq. (11). For illustrative purpose, let the thermal conductivity be a linear function of temperature, i.e., $f_k(U) = 1 + n_k U$ and let the surface temperature be the linear and exponential functions of the time represented respectively by $U_s = e^\tau$ and $U_s = 1 + 0.05\tau$. In this computation, the initial temperature is assumed to be zero and the coefficients of thermal conductivity and the heat capacity (n_k and m) are equal to -0.5 and 0.5, respectively. As can be seen from Figs. 1 and 2, the penetration depth moves faster when the parameter N decreases. This parameter represents the relative ratio of the energy transport due to conduction to that due to radiation. The radiative effect becomes important when the value of N is small. On the other hand, the problem is reduced to pure heat conduction without any radiation when N approaches infinity. The values of N in Figs. 1 and 2 cover a wide range within the practical interest in engineering. Numerical computations of temperature profile show that at any time the temperature at a given location increases as N decreases for the heating case. This is true physically because heat transfer rate increases with the inclusion of thermal radiation in the medium. It is found that the penetration depth at a given time increases when N decreases. This result is also expected physically, since the additional radiative heat flux always causes an increase of penetration depth.

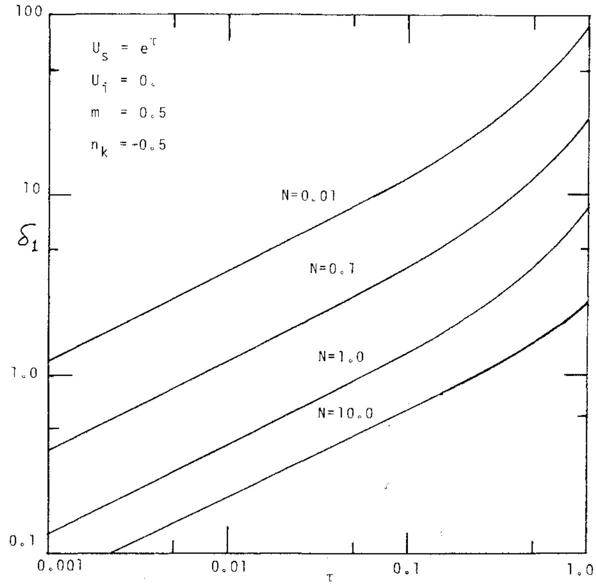

Fig. 1 Depth of penetration vs time for $U_s = e^\tau$.

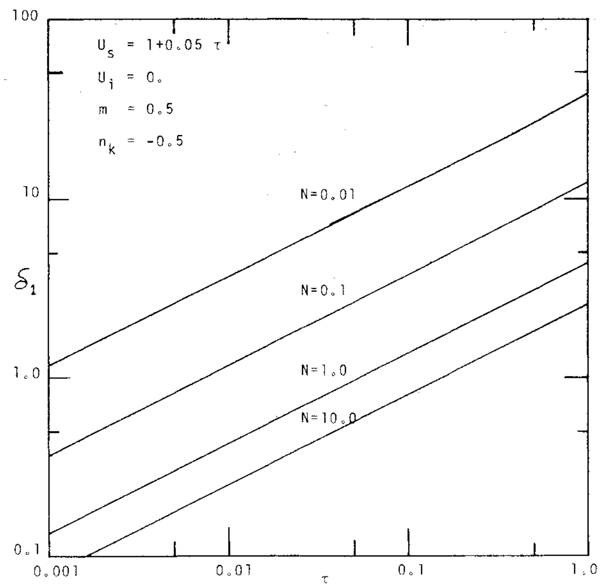

Fig. 2 Depth of penetration vs time for $U_s = 1 + 0.5\tau$.

Temperature distributions in the solid obtained from Eq. (17) with N as the parameter are shown in Fig. 3. The dashed line represents the case of constant thermal properties, i.e., $m = n_k = 0$; the solid line corresponds to the case of $m = 0.5$ and $n_k = -0.5$. The value of m chosen is reasonable for most solids[8] while that $n_k = \pm 0.5$ may exaggerate somewhat, although the same values have been used by Yang[9] for the pure transient heat conduction in a semi-infinite solid. In this computation, the initial temperature U_i and the surface temperature U_s are set equal to 0.25 and 1, respectively. It is found that the difference between the two solutions (constant property vs variable property) is about the same for different values of N. From Eq. (16) or (17), it is seen that the parameter N disappears when the surface temperature reaches absolute zero. This is due to the Rosseland's approximation for radiative heat flux. It should also be pointed out that the present solution which is based on an assumption of second order polynomial temperature profile is not applicable to the case with a periodic surface temperature. A periodic tem-

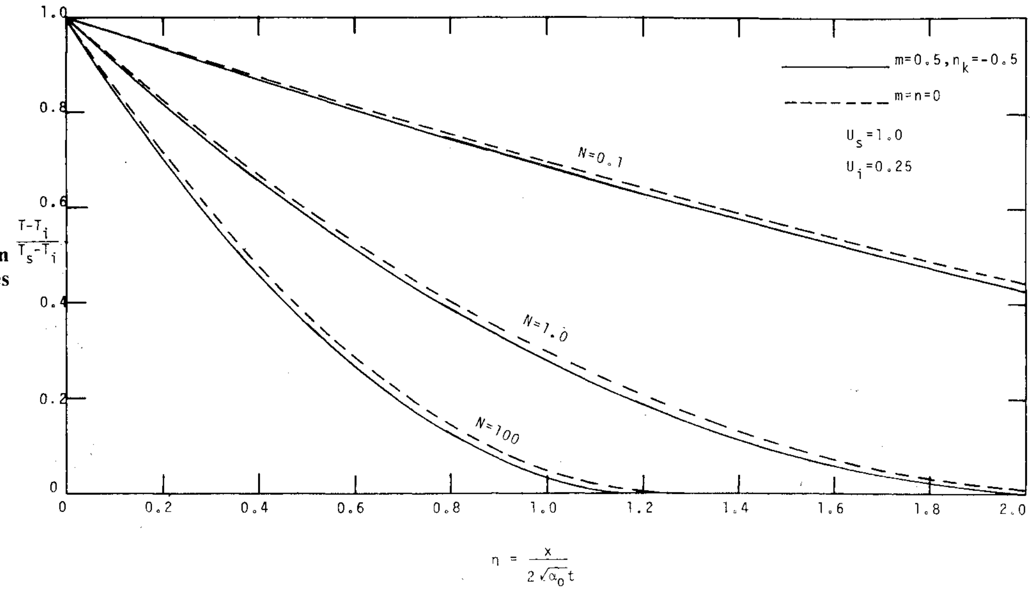

Fig. 3 Temperature distribution in a half space with various values of *N*.

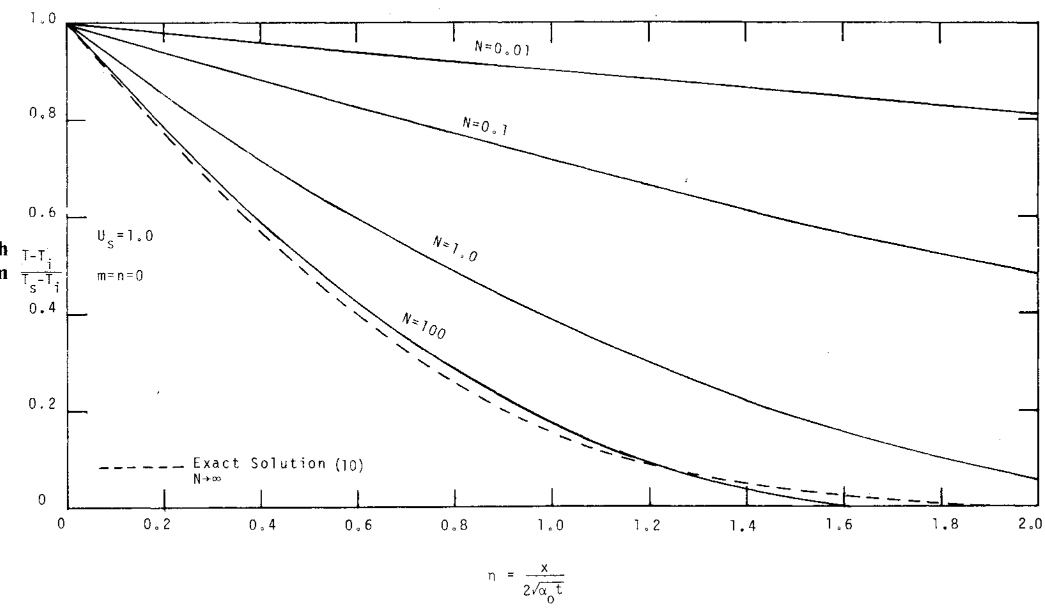

Fig. 4 Limiting solutions with constant properties and a uniform surface temperature.

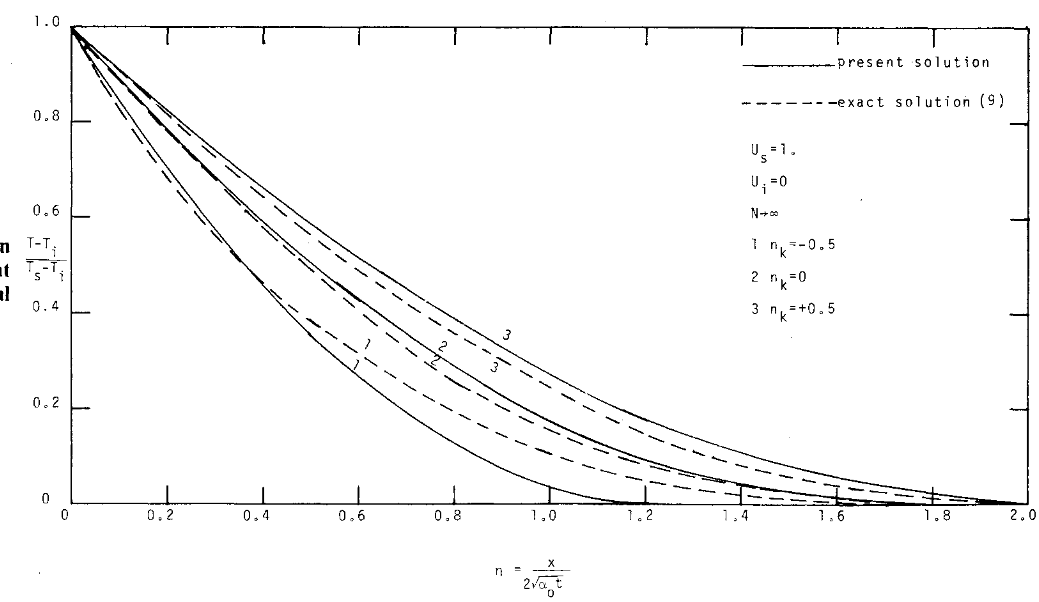

Fig. 5 Temperature distribution in a half space with a constant capacitance and a variable thermal conductivity.

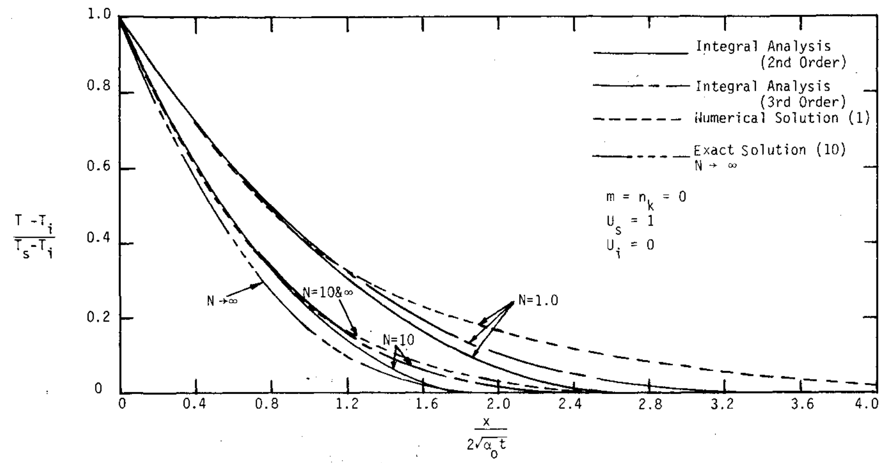

Fig. 6 Temperature distribution in a half space with $N=1.0$ and $N=10$.

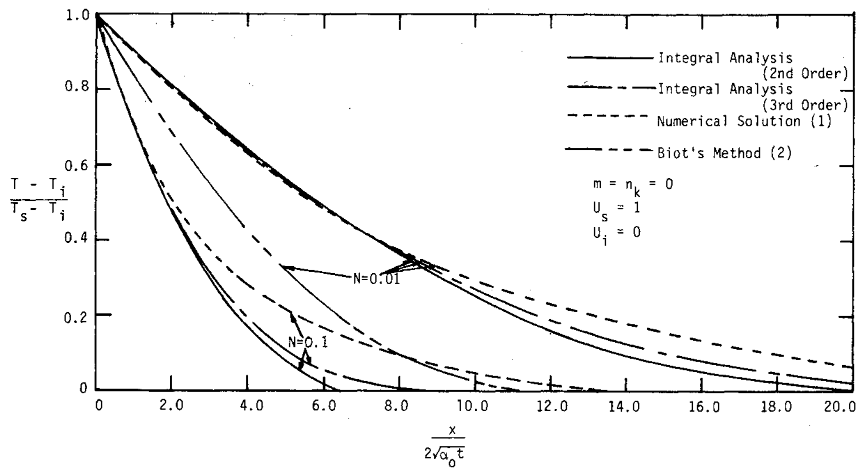

Fig. 7 Temperature distribution in a half space with $N=0.1$ and $N=0.01$.

perature profile in the solid should be assumed in obtaining the solution for this type of problem.

Figure 4 shows the temperature distribution in a semi-infinite solid with constant surface temperature ($U_s=1$) and constant thermal properties. For the purpose of comparison, the exact solution for the case of pure conduction,[10] i.e., $N\rightarrow\infty$ is also included. In this figure, the comparisons indicate a good agreement between the exact solution and the present limiting solution using $N=100$. This result further suggests that the radiative effect can be neglected when $N\geq 100$ for this particular problem.

Yang[9] applied a transformation first and then numerical integration to solve pure heat conduction in a semi-infinite solid with temperature dependent thermal conductivity and with a uniform surface temperature. It is of interest to compare his results with the present limiting solution with $N=100$. This comparison is presented in Fig. 5. It is seen that agreement between the present approximate and the exact solutions of Ref. 9 is reasonably good.

Similar to the use of integral approach in laminar boundary layer analysis, there is a question of uncertainty regarding the choice of temperature profiles. Özisik[10] has made a comment for the case of pure heat conduction that the results are rather insensitive as to the choice of the form of profile, a choice of higher order may improve the result slightly but adds more algebraic complexity. For practical purposes there is little gain in accuracy by choosing a profile higher than fourth degree. Whether this is true for a more general conduction problem remains to be seen. To investigate this point further, we consider a third order temperature profile of the form

$$T=T_i+(T_s-T_i)[1-(x/\delta)]^3 \tag{5b}$$

The above equation satisfies the previous three boundary conditions given by Eqs. (2b), (2d) and (2e) together with the fourth condition

$$\partial^2 T/\partial x^2=0 \quad \text{at} \quad x=\delta \tag{2f}$$

This condition is derived by evaluating Eq. (3) at $x=\delta$. Applying the same scheme mentioned earlier, we obtain an expression which is identical to Eq. (11) except with a different coefficient of $P(\tau)$ and $Q(\tau)$. They are

$$P(\tau)=\frac{\frac{1}{4}(1+mU_i)+(1/7)m(U_s-U_i)}{\frac{1}{2}[(1+mU_i)+(1/14)m(U_s-U_i)](U_s-U_i)}\frac{\mathrm{d}Us}{\mathrm{d}\tau} \tag{12b}$$

$$Q(\tau)=\frac{6[f_k(U_s)+(4/3N)U_s^3]}{\frac{1}{4}(1+mU_i)+(1/14)m(U_s-U_i)} \tag{13b}$$

In fact, the following more general form of temperature profile can be assumed

$$T-T_i=(T_s-T_i)[1-(x/\delta)]^p \tag{5c}$$

where p is an integer. This equation is obtained by using the conditions (2b), (2d), (2e) and (2f) plus the smooth boundary conditions. They are $\partial^3 T/\partial x^3=0\ldots\partial^{p-1}T/\partial x^{p-1}=0$ at $x=\delta$. The general solution is identical to Eq. (11) but $P(\tau)$ and $Q(\tau)$ are in terms of profile parameter p as

$$P(\tau)=\frac{\dfrac{1+mU_i}{1+p}+\dfrac{m(U_s-U_i)}{2p+1}}{\dfrac{1}{2}\left[\dfrac{1+mU_i}{1+p}+\dfrac{m(U_s-U_i)}{2(2p+1)}\right](U_s-U_i)}\frac{\mathrm{d}U_s}{\mathrm{d}\tau} \tag{12c}$$

$$Q(\tau) = \frac{2p\left[f_k(U_s) + \frac{4}{3N}U_s^3\right]}{\frac{1+mU_i}{1+p} + \frac{m(U_s - U_i)}{2(2p+1)}} \tag{13c}$$

Figure 6 presents a comparison of results based on third and second order temperature profile with a uniform surface temperature. The numerical solutions of Ref. 1 are replotted and rearranged in consistence with the present initial and boundary conditions. The case of constant properties and uniform surface temperature is considered. It is seen that the results with the third order temperature profile show a slight improvement over that of the second order profile. The numerical solutions of Ref. 1 indicate that the temperature curves with $N=10$ and $N\rightarrow\infty$ practically coincide. It appears that their results with $N\rightarrow\infty$ are somewhat higher than the exact solution of pure conduction in a semi-infinite solid (i.e., $N\rightarrow\infty$) which is also included in Fig. 6. For clarity, the results with $p>3$ are not shown in the figure. Our computations indicate that choice of higher order improve the temperature profile slightly at x near δ but has little effect upon the heat transfer rate at the wall.

Figure 7 depicts a similar comparison except for a pair of smaller values of N. Again, the present predicted temperature profile is consistently lower than that of Ref. 1. Overall, the agreement is reasonable from the engineering point of view. Lardners' solutions[2] based on Biot's variational method (after correction) are also included in this figure. It should be noted that Biot's variational method has been successfully employed to solve the various nonlinear transient heat conduction problems.[11-15] Apparently, the method has certain limitations in dealing with the combined conduction and radiation problem.

Conclusions

Transient temperature distribution in a gray, optically thick semi-infinite solid with temperature dependent properties and with an arbitrary time dependent surface temperature has been obtained based on the heat balance integral method. With the aid of this technique, the complex governing equation represented by a nonlinear partial differential equation is reduced to a first order initial value problem which is then solved exactly. The present results are compared with solutions available in the literature under limiting conditions including: 1) heat conduction with constant thermal properties, 2) heat conduction with variable thermal conductivity and 3) combined conduction and radiation with constant thermal properties. The good agreement further demonstrates that the heat balance integral technique provides a convenient and simple means of solving the complicated nonlinear heat transfer problems involving the interaction between the conduction and radiation.

References

[1] Heinish, R. P. and Viskanta, R., "Transient Combined Conduction-Radiation in Optically Thick Semi-infinite Medium," *AIAA Journal*, Vol. 6, July 1968, p. 1409.

[2] Lardner, T. J., "Approximate Solution of Combined Conduction-Radiation Problem," *AIAA Journal*, Vol. 7, Jan. 1969, p. 167.

[3] Goodman, T. R., "The Heat Balance Integral and Its Application to Problem Involving a Change of Phase," *ASME Transactions*, Vol. 80, Feb. 1958, p. 335.

[4] Reynolds, W. C. and Dolton, T. A., "The Use of Integral Methods in Transient Heat Transfer Analysis," ASME Paper 58-A-248, 1958.

[5] Schneider, P. J., "Radiation Cooling of Finite Heat-Conducting Solids," *Journal of the Aerospace Sciences*, Vol. 27, July 1960, p. 519.

[6] Chung, B.T.F. and Yeh, L. T., "Integral Method for Nonlinear Transient Heat Transfer in a Semi-infinite Solid," *Journal of Spacecraft and Rockets*, Vol. 10, Jan. 1973, p. 88.

[7] Rosseland, S., *Theoretical Astrophysics, Atomic Theory and the Analysis of Stillar Atmospheres and Envelopes*, Clarendon Press, Oxford, 1936.

[8] Goldsmith, A., Waterman, T. E., and Hirschhorn, H., *Handbook of Thermophysical Properties of Solid Materials*, Macmillian, N. Y., 1961.

[9] Yang, K. T., "Transient Conduction in a Semi-infinite Solid with Variable Thermal Conductivity," *Journal of Applied Mechanics*, Vol. 25, March 1958, p. 146.

[10] Özisik, M. N., *Boundary Value Problem of Heat Conduction*, International Textbook Company, Scranton, Pa., 1968.

[11] Lardner, T. J., "Biot's Variational Principle in Heat Conduction," *AIAA Journal*, Vol. 1, Jan. 1963, p. 196.

[12] Biot, M. A., *Variational Principle in Heat Transfer*, Clarendon Press, Oxford, 1970.

[13] Prasad, A. and Agrawal, H. C., "Biot's Variational Principle for Aerodynamic Ablation of Melting Solids," *AIAA Journal*, Vol. 12, Feb. 1974, p. 250.

[14] Chung, B.T.F. and Yeh, L. T., "Solidification and Melting of Materials Subjected to Convection and Radiation," *Journal of Spacecraft and Rockets*, Vol. 12, June 1975, p. 329.

[15] Chung, B.T.F. and Yeh, L. T., "Analysis of Heat Transfer in Slabs with Variable Properties Subjected to Radiative and Convective Boundary Conditions," ASME Paper WA/HT-67, 1975.